Exercise6F: Solutions

1 a $\boxed{P(n)}$

$$1+2+\cdots+n=\frac{n(n+1)}{2}$$

$\boxed{P(1)}$

If $n=1$ then

$$\text{LHS}=1$$

and

$$\text{RHS}=\frac{1(1+1)}{2}=1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1+2+\cdots+k=\frac{k(k+1)}{2}. \qquad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}\text{LHS of } P(k+1) &= 1+2+\cdots+k+(k+1)\\ &=\frac{k(k+1)}{2}+(k+1) \quad \text{(by (1))}\\ &=\frac{k(k+1)}{2}+\frac{2(k+1)}{2}\\ &=\frac{k(k+1)+2(k+1)}{2}\\ &=\frac{(k+1)(k+2)}{2}\\ &=\frac{(k+1)((k+1)+1)}{2}\\ &=\text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$$1+x+x^2+\cdots+x^n=\frac{1-x^{n+1}}{1-x}$$

$\boxed{P(1)}$

If $n=1$ then

$$\text{LHS}=1+x$$

and

$$\text{RHS}=\frac{(1-x^2)}{1-x}=\frac{(1-x)(1+x)}{1-x}=1+x.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 + x + x^2 + \cdots + x^k = \frac{1 - x^{k+1}}{1 - x}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
\text{LHS of } P(k+1) &= 1 + x + x^2 + \cdots + x^k + x^{k+1} \\
&= \frac{1 - x^{k+1}}{1 - x} + x^{k+1} \quad \text{(by (1))} \\
&= \frac{1 - x^{k+1}}{1 - x} + \frac{x^{k+1}(1 - x)}{1 - x} \\
&= \frac{1 - x^{k+1} + x^{k+1}(1 - x)}{1 - x} \\
&= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x} \\
&= \frac{1 - x^{k+2}}{1 - x} \\
&= \frac{1 - x^{(k+1)+1}}{1 - x} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1^2 - 1$$

and

$$\text{RHS} = \frac{1(1+1)(2+1)}{6} = 1.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
\text{LHS of } P(k+1) &= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{(by (1))} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}
\end{aligned}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$
$$= \frac{(k+1)(2k^2+7k+6)}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$
$$= \text{RHS of } P(k+1)$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$$1 \cdot 2 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = 1 \times 2 = 2$$

and

$$\text{RHS} = \frac{1 \times 2 \times 3}{3} = 2.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$1 \cdot 2 + \cdots + k \cdot (k+1) = \frac{k(k+1)(k+2)}{3}. \quad (1)$$

$\boxed{P(k+1)}$

$$\text{LHS of } P(k+1) = 1 \cdot 2 + \cdots + k \cdot (k+1) + (k+1) \cdot (k+2)$$
$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{(by (1))}$$
$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$$
$$= \frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$
$$= \frac{(k+1)(k+2)(k+3)}{3}$$
$$= \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$
$$= \text{RHS of } P(k+1)$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

e $\boxed{P(n)}$

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = \frac{1}{1 \times 3} = \frac{1}{3}$$

and

$$\text{RHS} = \frac{1}{2 \times 1 + 1} = \frac{1}{3}.$$

Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\frac{1}{1 \cdot 3} + \cdots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned}
\text{LHS of } P(k+1) &= \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \quad \text{(by (1))} \\
&= \frac{k(2k+3)}{(2k+1)(2k+3)} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{(2k+3)} \\
&= \frac{k+1}{(2(k+1)+1)} \\
&= \text{RHS of } P(k+1)
\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

f $\boxed{P(n)}$

$$\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

$\boxed{P(2)}$

If $n = 2$ then

$$\text{LHS} = 1 - \frac{1}{2^2} = \frac{3}{4}$$

and

$$\text{RHS} = \frac{2+1}{2 \times 2} = \frac{3}{4}.$$

Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$$

$\boxed{P(k+1)}$

$$\begin{aligned}\text{LHS of } P(k+1) &= \left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)\\ &= \frac{k+1}{2k}\left(1-\frac{1}{(k+1)^2}\right) \qquad \text{(by (1))}\\ &= \frac{k+1}{2k}\left(\frac{(k+1)^2}{(k+1)^2}-\frac{1}{(k+1)^2}\right)\\ &= \frac{k+1}{2k}\left(\frac{(k+1)^2-1}{(k+1)^2}\right)\\ &= \frac{(k+1)(k^2+2k)}{2k(k+1)^2}\\ &= \frac{k(k+1)(k+2)}{2k(k+1)^2}\\ &= \frac{(k+2)}{2(k+1)}\\ &= \frac{(k+1)+1}{2(k+1)}\\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

2 a $\boxed{P(n)}$

$11^n - 1$ is divisible by 10

$\boxed{P(1)}$

If $n = 1$ then

$$11^1 - 1 = 11 - 1 = 10$$

is divisible by 10. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$11^k - 1 = 10m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}11^{k+1}-1&=11\times 11^k-1\\&=11\times(10m+1)-1\quad\text{(by (1))}\\&=110m+11-1\\&=110m+10\\&=10(11m+1)\end{aligned}$$

is divisible by 10. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$3^{2n}+7$ is divisible by 8

$\boxed{P(1)}$

If $n=1$ then

$$3^{2\times 1}+7=9+7=16=2\times 8$$

is divisible by 8. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^{2k}+7=8m\qquad(1)$$

for some $k\in\mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}3^{2(k+1)}+7&=3^{2k+2}+7\\&=3^{2k}\times 3^2+7\\&=(8m-7)\times 9+7\quad\text{(by (1))}\\&=72m-63+7\\&=72m-56\\&=8(9m-7)\end{aligned}$$

is divisible by 8. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n\in\mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

7^n-3^n is divisible by 4

$\boxed{P(1)}$

If $n=1$ then

$$7^1-3^1=7-3=4$$

is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$7^k-3^k=4m\qquad(1)$$

for some $m\in\mathbb{Z}$.

$\boxed{P(k+1)}$

$$
\begin{aligned}
7^{k+1}-3^{k+1} &= 7\times 7^k - 3^{k+1}\\
&= 7\times(4m+3^k) - 3\times 3^k \quad \text{(by (1))}\\
&= 28m + 7\times 3^k - 3\times 3^k\\
&= 28m + 4\times 3^k\\
&= 4(7m+3^k)
\end{aligned}
$$

is divisible by 4. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$5^n + 6\times 7^n + 1$ is divisible by 4

$\boxed{P(1)}$

If $n = 1$ then

$$5^1 + 6\times 7^1 + 1 = 48 = 4\times 12$$

is divisible by 4. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$5^k + 6\times 7^k + 1 = 4m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$
\begin{aligned}
5^{k+1} + 6\times 7^{k+1} + 1 &= 5\times 5^k + 6\times 7\times 7^k + 1\\
&= 5\times(4m - 6\times 7^k - 1) + 42\times 7^{k+1}\\
&= 20m - 30\times 7^k - 5 + 42\times 7^k + 1\\
&= 20m + 12\times 7^k - 4\\
&= 4(5m + 3\times 7^k - 1)
\end{aligned}
$$

is divisible by 4. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 a $\boxed{P(n)}$

$4^n > 10\times 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4^4 = 256 \text{ and } \text{RHS} = 10\times 2^4 = 160.$$

Since $\text{LHS} > \text{RHS}$, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^k > 10\times 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$4^{k+1} > 10\times 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 4^{k+1} \\ &= 4 \times 4^k \\ &> 4 \times 10 \times 2^k \quad \text{(by (1))} \\ &= 40 \times 2^k \quad (\text{as } 10 > 2) \\ &= 20 \times 2^{k+1} \\ &> 10 \times 2^{k+1} \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Since $P(5)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$3^n > 5 \times 2^n$ where $n \geq 5$

$\boxed{P(5)}$

If $n = 5$ then

$$\text{LHS} = 3^5 = 243 \text{ and } \text{RHS} = 5 \times 2^5 = 160.$$

Since LHS $>$ RHS, $P(5)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k > 5 \times 2^k \text{ where } k \geq 5. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$3^{k+1} > 5 \times 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 3^{k+1} \\ &= 3 \times 3^k \\ &> 3 \times 5 \times 2^k \quad \text{(by (1))} \\ &= 15 \times 2^k \quad (\text{as } 10 > 2) \\ &> 10 \times 2^k \\ &= 5 \times 2^{k+1} \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Since $P(5)$ is true and $P(k+1)$ is true whenever $P(k)$ is true, $P(n)$ is true for all integers $n \geq 5$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$2^n > 2n$ where $n \geq 3$

$\boxed{P(3)}$

If $n = 3$ then

$$\text{LHS} = 2^3 = 8 \text{ and } \text{RHS} = 2 \times 3 = 6.$$

Since LHS $>$ RHS, $P(3)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$2^k > 2k \text{ where } k \geq 3. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$2^{k+1} > 2(k+1).$$

$$\begin{aligned}\text{LHS of } P(k+1) &= 2^{k+1} \\ &= 2 \times 2^k \\ &> 2 \times 2k \quad \text{(by (1))} \\ &= 4k \\ &= 2k + 2k \\ &\geq 2k + 2 \quad (\text{as } 2k \geq 2) \\ &= 2(k+1) \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all integers $n \geq 3$ by the principle of mathematical induction.

d $\boxed{P(n)}$

$n! > 2^n$ where $n \geq 4$

$\boxed{P(4)}$

If $n = 4$ then

$$\text{LHS} = 4! = 24 \text{ and RHS} = 2^4 = 16.$$

Since LHS $>$ RHS, $P(4)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$k! > 2^k \text{ where } k \geq 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$(k+1)! > 2^{k+1}.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= (k+1)! \\ &= (k+1)k! \\ &> (k+1) \times 2^k \quad \text{(by (1))} \\ &> 2 \times 2^k \quad (\text{as } k+1 > 2) \\ &= 2^{k+1} \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all integers $n \geq 4$ by the principle of mathematical induction.

4 **a** $\boxed{P(n)}$

$a_n = 2^n + 1$

$\boxed{P(1)}$

If $n = 1$ then

$\text{LHS} = a_1 = 3 \text{ and RHS} = 2^1 + 1 = 3.$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 2^k + 1. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 2^{k+1} + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k - 1 \quad \text{(by definition)} \\ &= 2(2^k + 1) - 1 \quad \text{(by (1))} \\ &= 2^{k+1} + 2 - 1 \\ &= 2^{k+1} + 1 \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b $\boxed{P(n)}$

$a_n = 5^n - 1$

$\boxed{P(1)}$

If $n = 1$ then

$\text{LHS} = a_1 = 4 \text{ and RHS} = 5^1 - 1 = 4.$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 5^k - 4. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 5^{k+1} - 4.$$

$$\begin{aligned}\text{LHS} &= a_{k+1} \\ &= 5a_k + 4 \quad \text{(by definition)} \\ &= 5(5^k - 1) + 4 \quad \text{(by (1))} \\ &= 5^{k+1} - 5 + 4 \\ &= 5^{k+1} - 1 \\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

c $\boxed{P(n)}$

$a_n = 2^n + n$

$\boxed{P(1)}$

If $n = 1$ then

$\text{LHS} = a_1 = 3 \text{ and RHS} = 2^1 + 1 = 3.$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$a_k = 2^k + k. \qquad (1)$$

$\boxed{P(k+1)}$

We have to show that

$$a^{k+1} = 2^{k+1} + k + 1.$$

$$\begin{aligned}\text{LHS of } P(k+1) &= a_{k+1} \\ &= 2a_k - k + 1 \quad \text{(by definition)} \\ &= 2(2^k + k) - k + 1 \quad \text{(by (1))} \\ &= 2^{k+1} + 2k - k + 1 \\ &= 2^{k+1} + k + 1 \\ &= \text{RHS of } P(k+1)\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

5 $\boxed{P(n)}$

3^n is odd where $n \in \mathbb{N}$

$\boxed{P(1)}$

If $n = 1$ then clearly

$$3^1 = 3$$

is odd. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$3^k = 2m + 1 \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}3^{k+1} &= 3 \times 3^k \\ &= 3 \times (2m+1) \quad \text{(by (1))} \\ &= 6m + 3 \\ &= 6m + 2 + 1 \\ &= 2(3m+1) + 1\end{aligned}$$

is odd, so that $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 **a** $\boxed{P(n)}$

$n^2 - n$ is even, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^2 - \times 1 = 0$$

is even. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^2 - k$ is even. Therefore,

$$k^2 - k = 2m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned}(k+1)^2 - (k+1) &= k^2 + 2k + 1 - k - 1 \\ &= k^2 + k \\ &= (k^2 - k) + 2k \\ &= 2m + 2k \quad \text{(by (1))} \\ &= 2(m + k)\end{aligned}$$

Since this is even, $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives

$$n^2 - n = n(n-1).$$

As this is the product of two consecutive numbers, one of them must be even, so that the product will also be even.

7 **a** $\boxed{P(n)}$

$n^3 - n$ is divisible by 3, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$1^3 - 1 = 0$$

is divisible by 3. Therefore, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that $k^3 - k$ is divisible by 3. Therefore,

$$k^3 - k = 3m \qquad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

We have to show that $(k+1)^3-(k+1)$ is divisible by 3.

$$\begin{aligned}(k+1)^3-(k+1) &= k^3+3k^2+3k+1-k-1\\ &= k^3-k+3k^2+3k\\ &= (k^3-k)+3k^2+3k\\ &= 3m+3k^2+3k \quad \text{(by (1))}\\ &= 3(m+k^2+k)\end{aligned}$$

Since this is divisible by 3, $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

b Factorising the expression gives $n^3+n=n(n^2-1)=n(n-1)(n+1)$.
As this is the product of three consecutive numbers, one of them must be divisible by 3, so that the product will also be divisible by 3.

8 1

n	1	2	3	4	5
a_n	9	99	999	9999	99999

2 We claim that $a_n = 10^n - 1$.

3 $\boxed{P(n)}$

$a_n = 10^n - 1$

$\boxed{P(1)}$

If $n = 1$, then

$\text{LHS} = a_1 = 9 \text{ and RHS} = 10^1 - 1 = 9.$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$a_k = 10^k - 1. \qquad (1)$

$\boxed{P(k+1)}$

We have to show that

$a^{k+1} = 10^{k+1} - 1.$

$$\begin{aligned}\text{LHS} &= a_{k+1}\\ &= 10a_k + 9 \quad \text{(by definition)}\\ &= 10(10^k-1)+9 \quad \text{(by (1))}\\ &= 10^{k+1}-10+9\\ &= 10^{k+1}-1\\ &= \text{RHS}\end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

9 1

n	1	2	3	4	5	6	7	8	9	10
f_n	1	1	2	3	5	8	13	21	34	55

2 $\boxed{P(n)}$

$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_3 - 1 = 2 - 1 = 1.$$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$f_1 + f_2 + \cdots + f_k = f_{k+2} - 1. \quad (1)$

$\boxed{P(k+1)}$

$$\begin{aligned} \text{LHS of } P(k+1) &= f_1 + f_2 + \cdots + f_k + f_{k+1} \\ &= f_{k+2} - 1 + f_{k+1} \quad \text{(by (1))} \\ &= f_{k+1} + f_{k+2} - 1 \\ &= f_{k+3} - 1 \quad \text{(by definition)} \\ &= f_{(k+1)+2} - 1 \\ &= \text{RHS of } P(k+1) \end{aligned}$$

Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

3 $f_1 = 1$
$f_1 + f_3 = 1 + 2 = 3$
$f_1 + f_3 + f_5 = 3 + 5 = 8$
$f_1 + f_3 + f_5 + f_7 = 8 + 13 = 21$

4 From the pattern observed above, we claim that
$f_1 + f_3 + \ldots + f_{2n-1} = f_{2n}$.

5 $\boxed{P(n)}$

$f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

$\boxed{P(1)}$

If $n = 1$ then

$$\text{LHS} = f_1 = 1$$

and

$$\text{RHS} = f_2 - 1 = 2 - 1 = 1.$$

Since $\text{LHS} = \text{RHS}$, $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}. \quad (1)$$

$\boxed{P(k+1)}$

$$\begin{aligned} \text{LHS} &= f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} \\ &= f_{2k} + f_{2k+1} \quad \text{(by (1))} \\ &= f_{2k+2} \quad \text{(by definition)} \end{aligned}$$

$= f_{2(k+1)}$
$= \text{RHS}$
Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

6 $\boxed{P(n)}$

The Fibonacci number f_{3n} is even.

$\boxed{P(1)}$

If $n = 1$ then

$$f_3 = 2$$

is even, therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that f_{3k} is even. That is,

$$f_{3k} = 2m \quad (1)$$

for some $m \in \mathbb{Z}$.

$\boxed{P(k+1)}$

$$\begin{aligned} f_{3(k+1)} &= f_{3k+3} \\ &= f_{3k+2} + f_{3k+1} \quad \text{(by definition)} \\ &= f_{3k+1} + f_{3k} + f_{3k+1} \\ &= 2f_{3k+1} + f_{3k} \\ &= 2f_{3k+1} + 2m \quad \text{(by (1))} \\ &= 2(f_{3k+1} + m) \end{aligned}$$

Since this is even, $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

10 $\boxed{P(n)}$

Since we're only interested in odd numbers our proposition is:
$4^{2n-1} + 5^{2n-1}$ is divisible by 9, where $n \in \mathbb{N}$.

$\boxed{P(1)}$

If $n = 1$ then

$$4^1 + 5^1 = 9$$

is divisible by 9. Therefore $P(1)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true so that

$$4^{2k-1} + 5^{2k-1} = 9m \qquad (1)$$

for some $k \in \mathbb{Z}$.

$\boxed{P(k+1)}$

The next odd number will be $2k+1$.

Therefore, we have to prove that

$$4^{2k+1}+5^{2k+1}$$

is divisible by 9.

$$\begin{aligned}4^{2k+1}+5^{2k+1} &= 4^2\times 4^{2k-1}+5^2\times 5^{2k-1}\\ &= 16\times(9m-5^{2k-1})+25\times 5^{2k-1} \quad \text{(by (1))}\\ &= 144m-16\times 5^{2k-1}+25\times 5^{2k-1}\\ &= 144m+9\times 5^{2k-1}\\ &= 9(16+5^{2k-1})\end{aligned}$$

Since this is divisible by 9, we've shown that $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

11 $P(n)$

A set of numbers S with n numbers has a largest element.

$P(1)$

If $n=1$, then set S has just one element. This single element is clearly the largest element in the set.

$P(k)$

Assume that $P(k)$ is true. This means that a set of numbers S with k numbers has a largest element.

$P(k+1)$

Suppose set S has $k+1$ numbers. Remove one of the elements, say x, so that we now have a set with k numbers. The reduced set has a largest element, y. Put x back in set S, so that its largest element will be the larger of x and y. Therefore $P(k+1)$ is true.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

12 $P(n)$

It is possible to walk around a circle whose circumference includes n friends and n enemies (in any order) without going into debt.

$P(1)$

If $n=1$, there is one friend and one enemy on the circumference of a circle. Start your journey at the friend, receive \$1, then walk around to the enemy and lose \$1. At no point will you be in debt, so $P(1)$ is true.

$P(k)$

Assume that $P(k)$ is true. This means that it is possible to walk around a circle with k friends and k enemies (in any order) without going into debt, provided you start at the correct point.

$P(k+1)$

Suppose there are $k+1$ friends and $k+1$ enemies located on the circumference of the circle, in any order. Select a friend whose next neighbour is an enemy (going clockwise), and remove these two people. As there are now k friends and k enemies, it is possible to walk around the circle without going into debt, provided you start at the correct point. Now reintroduce the two people, and start walking from the same point. For every part of the journey you'll have the same amount of money as before except when you meet the added friend, who gives you \$1, which is immediately lost to the added enemy.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

13 $P(n)$

Every integer j such that $2 \leq j \leq n$ is divisible by some prime.

$\boxed{P(2)}$

If $n = 2$, then $j = 2$ is clearly divisible by a prime, namely itself. Therefore $P(2)$ is true.

$\boxed{P(k)}$

Assume that $P(k)$ is true. Therefore, every integer j such that $2 \leq j \leq k$ is divisible by some prime.

$\boxed{P(k+1)}$

We need to show that integer j such that $2 \leq j \leq k+1$ is divisible by some prime. By the induction assumption, we already know that every j with $2 \leq j \leq k$ is divisible by some prime. We need only prove that $k+1$ is divisible by a prime. If $k+1$ is a prime number, then we are finished. Otherwise we can find integers a and b such that $k+1 = ab$ and $2 \leq a \leq k$ and $2 \leq b \leq k$. By the induction assumption, the number a will be divisible by some prime number. Therefore $k+1$ is divisible by some prime number.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.

14 If such a colouring of the regions is possible we will call it a **satisfactory colouring** .

$\boxed{P(n)}$

If n lines are drawn then the resulting regions have a satisfactory colouring.

$\boxed{P(1)}$

If $n = 1$, then there is just one line. We colour one side black and one side white. This is a satisfactory colouring. Therefore $P(1)$ is true.

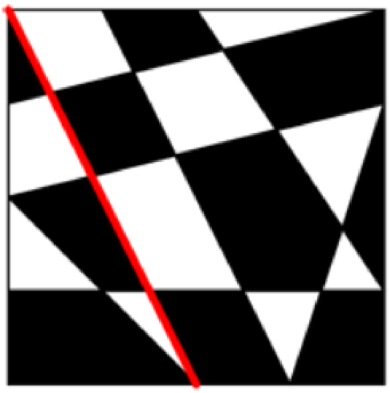
$\boxed{P(k)}$

Assume that $P(k)$ is true. This means that we can obtain a satisfactory colouring if there are k lines drawn.

$\boxed{P(k+1)}$

Now suppose that there are $k+1$ lines drawn. Select one of the lines, and remove it. There are now k lines, and the resulting regions have a satisfactory colouring since we assumed $P(k)$ is true. Now add the removed line. This will divide some regions into into two new regions with the same colour, so this is not a satisfactory colouring.

However, if we switch each colour on **one** side of the line we obtain a satisfactory colouring.



This is because inverting a satisfactory colouring will always give a satisfactory colouring, and regions separated the new line will not have the same colour.
Therefore $P(n)$ is true for all $n \in \mathbb{N}$ by the principle of mathematical induction.